5. Homework Assignments

Dynamical Systems II

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http://dynamics.mi.fu-berlin.de/lectures/

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Problem 1: Consider the parallel flow $\dot{x} = \alpha > 0$, $\dot{y} = \beta$ with irrational slope β/α on the standard 2-torus. Any line with rational (or infinite) slope q/p defines a global Poincaré section. Determine the rotation number of the Poincaré map.
Hint: Consider co-prime integers p and q and use $pp' + qq' = 1$ for suitable integers p' and q'.

Problem 2: Consider the general pendulum equation

$$\ddot{x} + \nabla V(x) \;=\; 0, \qquad x \in \mathbb{R}^N,$$

with smooth potential $V : \mathbb{R}^N \to \mathbb{R}$. Let $x = \dot{x} = 0$ be a *hyperbolic* equilibrium.

Consider the Hamiltonian $H(x, \dot{x}) = \frac{1}{2}\|\dot{x}\|^2 + V(x)$. Prove or disprove: Locally, the level set of the equilibrium is the union of its stable and unstable manifolds, i.e.

$$W^{\mathrm{u}}_{\mathrm{loc}}(0,0) \cup W^{\mathrm{s}}_{\mathrm{loc}}(0,0) \;=\; \{(x,\dot{x})\ ;\ H(x,\dot{x}) = H(0,0)\}_{\mathrm{loc}}$$

(i) for one degree of freedom, $N = 1$;

(ii) for more degrees of freedom, $N > 1$.

Problem 3: A real number x is a *Diophantine number* if for every $\epsilon > 0$ there exists $c_\epsilon > 0$ such that $|x - \frac{p}{q}| \geq c_\epsilon/|q|^{2+\epsilon}$ for every rational number p/q. A subset of $\mathbb{R}$ is called *generic* if it contains a countable intersection of sets, each of which is open and dense.

(i) Show that the set of Diophantine numbers in $[0, 1]$ has full Lebesgue measure 1.

(ii) Show that the set of non-Diophantine numbers is generic.

Problem 4: Consider the Fibonacci-Iteration

$$x_{n+1} = x_n + x_{n-1} \pmod 1$$

on the 2-torus $(x_n, x_{n-1}) \in (\mathbb{R}/\mathbb{Z})^2$, i.e., consider only the noninteger fractional part of the iteration. Is it well defined? Calculate stable and unstable manifolds of the fixed point $(0, 0)$ under the iteration Ψ^n. Are they dense on the 2-torus? Are the homoclinic orbits of $(0, 0)$ dense?